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PK 12

Addressing the Hammer-and-Nail Phenomenon

Our human tendency is to approach a problem using a familiar tool instead of analyzing the problem. Here are pedagogical suggestions to help students minimize their mathematical impulsivity, cultivate an analytic disposition, and develop conceptual understanding.

Kien H. Lim

Solve this story problem and then take a moment to think about the problem and anticipate how your students would solve it.

> A candle is burning at a constant rate. When it has burned 30 mm, its height is 75 mm. When it has burned 60 mm, what is the candle's height?

Figure 1 shows three solutions, recorded using a Livescribe pen, from ninth graders taking a geometry course. Why do you think the students solve the problem in these ways? How are their solutions different?

In the first solution (see figure 1a), the student set up a proportion and used cross multiplication to obtain an answer of 150. In figure 1b, the student knew that the

Fig. 1

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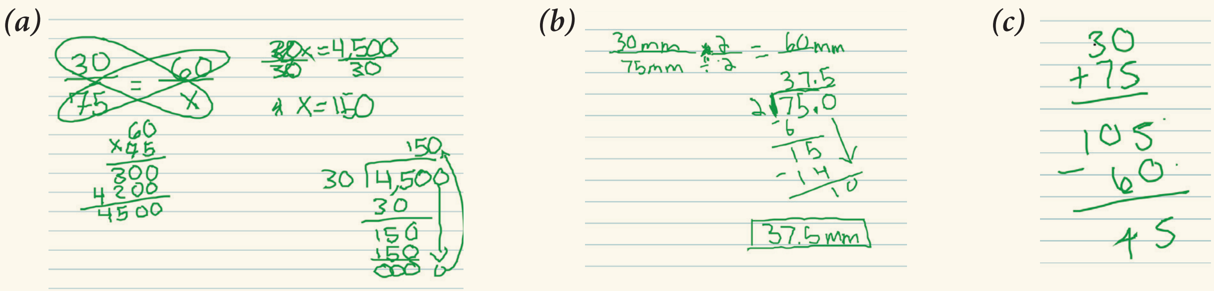

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Ninth-grade geometry students had three different solutions to the Candle problem: (a) Assuming that the candle's height and burned length are directly proportional, this student set up a proportion and used cross multiplication to obtain an answer of 150; (b) this student knew that the height had to decrease as the candle burned more of its length, so he divided the height of 75 mm by 2 instead of multiplying by 2, an inversely proportional relationship. In (c), the student found the initial height to be 105 and then subtracted 60 to obtain an answer of 45.

height had to decrease as the candle burned more of its length, so he divided the height of 75 mm by 2 instead of multiplying by 2. Whereas the first solution is based on the assumption that the candle's height and burned length are directly proportional, the second solution is based on an inversely proportional relationship. In the third solution (see figure 1c), the student found the initial height to be 105 and then subtracted 60 to obtain an answer of 45.

When this missing-value problem (values of three quantities are given to find the value of the fourth quantity) was tested with 29 preservice teachers, 15 of them wrote 150 mm as the answer and only 6 reasoned additively to obtain the correct answer of 45 mm. These results highlight students' tendency to use a proportion (hammer) to solve a missing-value problem even when it is nonproportional (mistaken as a nail); this is called the *hammer-and-nail phenomenon*.

The hammer-and-nail metaphor appeared in the literature as a researcher's tendency in conducting scientific research: "It is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail" (Maslow 1966, pp. 15–16). In the context of teaching and learning mathematics, making the hammer-and-nail phenomenon explicit can increase educators' awareness of students' tendency to rely on familiar procedures and ideas instead of taking time to think and analyze a problem situation. This article is written with mathematics educators and teachers in mind to (1) illustrate examples of the hammer-and-nail phenomenon in solving mathematical problems, (2) present two plausible causes for this phenomenon, and (3) offer pedagogical strategies for addressing it.

MANIFESTATIONS OF THE HAMMER-AND-NAIL PHENOMENON

The hammer-and-nail phenomenon is noticeable when an error or an inefficient solution is observed to arise from inappropriately applying a tool (hammer) to solve a mathematical problem (nail). It is more likely to occur among students who are *tool-oriented* than students who are *situation-oriented*. Whereas a situation-oriented student analyzes a problem and then decides on a strategy

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to solve it, a tool-oriented student tends to approach a problem with the tool with which the student is familiar (e.g., see figure 1a). Tools that are commonly and inappropriately used by students include algorithms (e.g., finding a common denominator), problem-solving strategies (e.g., generalizing solely on the basis of a numerical pattern), formulas (e.g., a Pythagorean equation), and intuitions (e.g., multiplication leads to a larger value).

The hammer-and-nail phenomenon is rather common among students. Lim (2012) reported that more than 50 percent of 307 preservice K–8 teachers found a person's traveling speed from a line graph using the $s = d/t$ formula (hammer) without realizing that the y-value depicted on the graph refers to the person's distance from home (nail) instead of the distance traveled in the first 20 minutes (see figure 2). Only 18 percent chose the correct answer by finding the slope of the line, that is, $s = \Delta d/\Delta t$. If the graph represents a directly proportional relationship—that is, the vertical axis represents the distance from Gina's friend's house and the line passes through the origin—then the $s = d/t$ formula would have been appropriate.

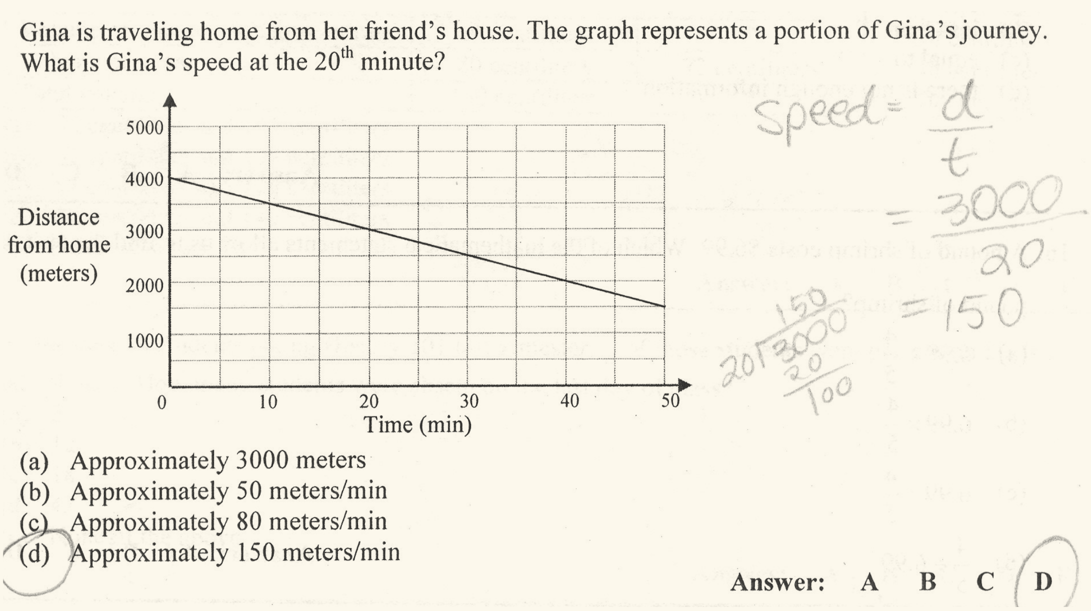
The more familiar one is with a particular tool, the more likely one is to use it. Van Dooren and colleagues (2005) investigated the misapplication of proportional reasoning in solving missing-value problems among 1,062 students from grades 2 to 8. They found that second and third graders were mainly dominated by additive reasoning, fourth through sixth graders were mainly dominated by proportional reasoning (they tended to choose proportional answers for nonproportional missing-value problems), and seventh and eighth graders were more likely to choose a solution strategy based on an appropriate model for the problem. These results suggest students are most susceptible to overgeneralizing proportional reasoning in the fourth through sixth grades when they are learning and practicing strategies for solving proportional problems. As for the candle problem in figure 1, the author conjectures that the number of students who used a proportion to solve it would have been lower if they were to create a mental model of the burning candle.

TWO PLAUSIBLE CAUSES

Several factors could contribute to the ubiquity of the hammer-and-nail phenomenon among mathematics students. Two explanations are offered here: (1) a dual-process theory of cognition and (2) traditional methods of teaching school mathematics.

Several researchers in cognitive psychology have posited that two distinct cognitive systems of reasoning

Fig. 2



A student misused the speed = distance/time formula.

exist: “System 1 processes are rapid, parallel, and automatic in nature: Only their final product is posted in consciousness,” whereas “system 2 thinking is slow and sequential in nature and makes use of the central working memory system” (Evans 2006, p. 454). The two systems function differently, each specializing at different kinds of tasks, but often work cooperatively (Sloman 1996). At other times, however, they may compete, each trying to generate a response. Because of its speed and efficiency, system 1 often precedes and overshadows system 2. System 2 may kick in to override the response of system 1. In the context of solving math problems, system 1 dominates when a student exhibits *impulsive anticipation*—spontaneously proceeding with an action that comes to mind without analyzing the problem situation or considering the relevance of the anticipated action to the problem situation. System 2 dominates when a student exhibits *analytic anticipation*—analyzing the problem situation and establishing a goal or a criterion to guide one’s actions (Lim 2006).

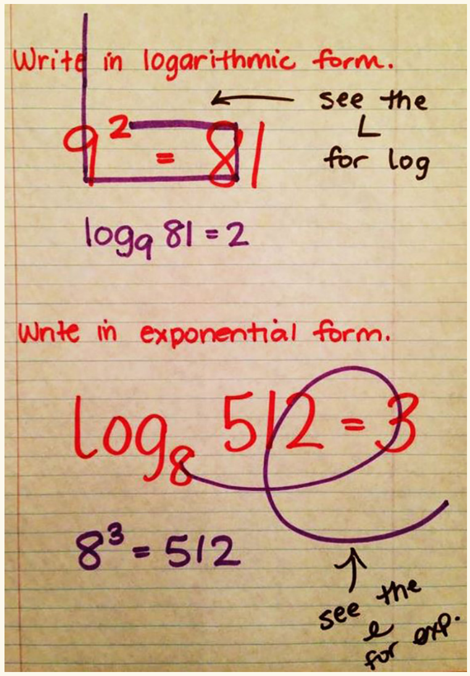
When mathematics is taught with too much emphasis on skill and procedure, system 1 thinking tends to be strengthened. Stigler and Hiebert (1999) reported that more than 60 percent of teachers in the United States described skills as the main thing that they wanted their students to learn. To improve student test performance, some teachers teach their students to use key words (e.g., *of* means multiply), acronyms (e.g., SOH-CAH-TOA), shortcuts (e.g., the invert-multiply strategy), and tricks (e.g., figure 3 shows the use of *L*-signature and *e*-signature as a way to remember converting equations between exponential and logarithmic forms). Such tips reinforce the dominance of system 1 and suppress system 2 because students are relieved from having to grapple with the mathematics they are learning and to analyze the problem they are solving. For example, when asked to solve $3^{2x} = 27$, a student may apply the *e*-signature to obtain $\log_3 27 = 2x$ and

$$x = \frac{\log_3 27}{2}$$

instead of thinking about what value of x would make 3^{2x} equal to 27 and predicting that x could be 1.5 because 3^3 equals 27.

By being aware of the hammer-and-nail phenomenon, teachers can minimize teaching in a manner that accentuates speed over understanding. In addition, teachers can explicitly help students recognize their hammer-and-nail tendencies by having them work on certain mathematical problems that are designed to elicit conceptual errors.

Fig. 3



Help students remember how to convert between the two forms of equation. (Source: http://schooloffisher.blogspot.com/2013/12/exponential-logarithmic-form.html?m=1)

USING SS-SD PROBLEMS TO ADDRESS THE HAMMER-AND-NAIL PHENOMENON

When students observe instances of their impulsivity and overreliance on *instrumental understanding* (i.e., rules without reason), they can better appreciate the importance of being analytic and having *relational understanding*—knowing both what to do and why (Skemp 1976). Teachers can incorporate problems that are *superficially similar but structurally different* (SS-SD) to elicit errors due to impulsive actions and thereby can uncover the conceptual structures underlying the superficial similarity.

SS-SD problems are those that share similar surface features but involve different mathematical properties, relationships, or structures. The following pair of statements exemplifies the fact that superficially similar statements can have different truth-values; the first statement involves absolute comparison, whereas the second statement involves relative comparison.

Determine whether the following statements are true or false.

1. If Jorge has four marbles more than Maria, then Maria has four fewer marbles than Jorge.
2. If George has one-fourth times more marbles than Mary, then Mary has one-fourth times fewer marbles than George.

The first statement is true because if $A = B + 4$, then $B = A - 4$, but the second statement is false because if $A = B + (1/4)B$, then $A = (5/4)B$, which means $B = (4/5)A$, and therefore B cannot be equal to $A - (1/4)A$ (assuming $A > 0$ and $B > 0$). This task, when implemented effectively, can help students learn the difference between (1) additive comparison, which uses subtraction to find the absolute difference between two quantities, and (2) multiplicative comparison, which uses division to find the relative difference (i.e., ratio) between two quantities. The second statement allows the teacher to draw students' attention to the referent amount for "one-fourth times."

Figure 4 presents the solutions to a pair of SS-SD problems. Although both problems could be solved correctly using algebra or a strip diagram, this particular solution strategy is appropriate for the first problem but not for the second problem. This Split and Adjust strategy begins by tentatively assuming both children have the same amount and then adjusting according to the difference stated in the word problem. Adjusting each child's number of marbles by adding and subtracting 15 marbles (since they have a net difference of 30 marbles) works well for the Jorge-Maria problem. In a class assessment, 25 preservice mathematics teachers of grades 4–8 were asked to explain the solution to the Jorge-Maria problem and then solve the George-Mary problem with the aid of a strip diagram. Five preservice teachers used the Split and Adjust strategy to solve the George-Mary problem by taking 1/3 of 60 marbles instead of 1/3 of Mary's number of marbles. The author used this mistake to draw students' attention to the referent amount for 1/3, an important habit of mind in dealing with fractions.

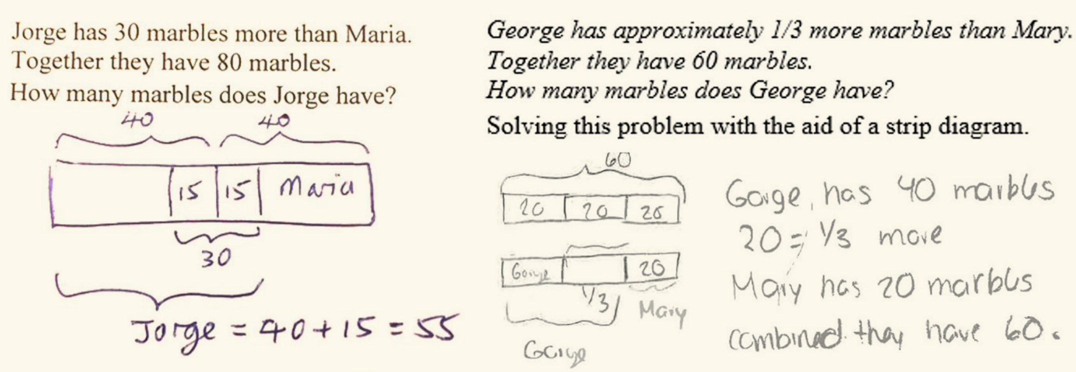
Here is a sequence of three somewhat similar problems.

The Handshake Problem
If each student in a class of 35 shakes the hand of each classmate exactly once, how many handshakes are there altogether?

Single Elimination Problem
There are 16 players in a chess tournament. A player is knocked out of the competition once he loses a game. How many games are needed to determine the champion (assuming that there are no ties)?

Round Robin Problem
There are 16 players in a chess tournament. Each player must play against all other players. How many games will be played altogether (assuming that there are no ties)?

Fig. 4



Although both problems could be solved correctly using algebra or a strip diagram, this particular solution strategy is appropriate for the first problem but not for the second problem.

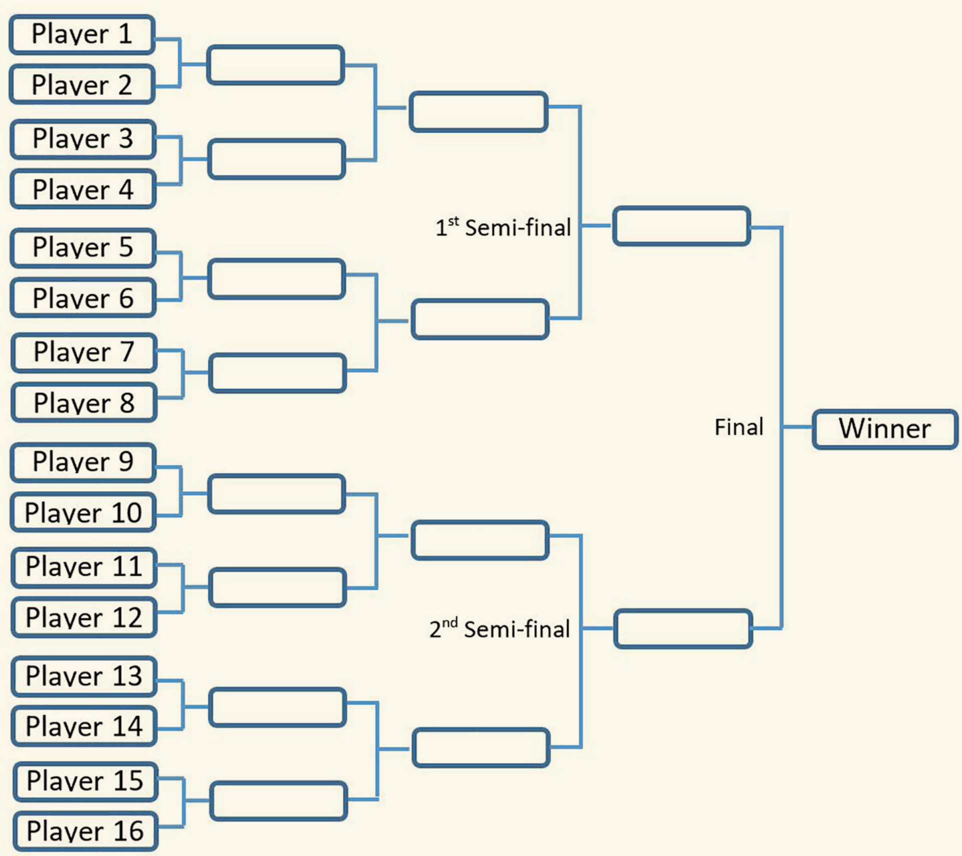
For the Handshake problem, the total number of handshakes is (35 × 34)/2 because each student must shake hands with the other 34 students and the product is divided by two to compensate for double counting; student A shaking student B's hand and student B shaking student A's hand should be counted as one handshake. For the Round Robin problem, the total number of games is (16 × 15)/2 for the same reason. Hence, the Handshake problem and the Round Robin problem are structurally equivalent to one another, but they are structurally different from the Single Elimination problem—the total number of games needed is 15 because one player is knocked out in each game (see figure 5) and the winner will emerge on the 15th game, where the second best player is knocked out. When implemented effectively, such SS-SD problems can help establish a classroom culture where students know that they are expected to analyze and make sense of the problems instead of blindly applying a recently learned solution technique.

We advocate the use of SS-SDs to help students address their hammer-and-nail tendency. In using SS-SDs, teachers can emphasize *quantitative reasoning* (Thompson 1993), reinforce *attendance to meaning* (Harel, Fuller, and Rabin 2009, p. 2008), and assess for conceptual understanding.

Emphasizing Quantitative Reasoning

Gaining awareness of mathematical impulsivity does not guarantee a correct solution. In solving a word problem, a student needs to identify the quantities—attributes that can be measured or amounts that can be counted—and establish meaningful relationships among them, as opposed to merely circling the numbers stated in the problem and then deciding what to do with those numbers. Thompson (1993) defines

Fig. 5



This is a template for a single-elimination tournament.

quantitative reasoning as "the analysis of a situation into a quantitative structure—a network of quantities and relationships" (p. 165). Students who engage in quantitative reasoning are more likely to be situation-oriented because they would analyze the problem situation in terms of quantities and relationship. On the other hand, tool-oriented students are less likely to engage in quantitative reasoning.

One way to foster quantitative reasoning is to challenge students to explain the mathematical structure underlying a problem situation. For example, when students set up a proportion $a/b = c/x$ to solve a missing-value problem, they should provide "reasons in support of claims made about the structural relationships among four quantities" (Lamon 2007, p. 638). Lim (2009) used both proportional problems and nonproportional problems to highlight the importance of analyzing the problem situation, determining the covarying quantities, and identifying the invariant relationship. Whereas quantities in a proportional missing-value problem are related by an invariant ratio, $a/b = c/x$, the quantities depicted in a nonproportional missing-value problem may be related by an invariant product ($ab = cx$), an invariant sum ($a + b = c + x$), or an invariant difference ($a - b = c - x$). Below is a pair of missing-value problems (adapted from Lim 2009) to highlight the difference between a proportional problem, which involves an invariant ratio, and a nonproportional problem that involves an invariant difference.

> Whereas a situation-oriented student analyzes a problem and then decides on a strategy to solve it, a tool-oriented student tends to approach a problem with the tool with which the student is familiar.

Proportional Problem (Invariant Ratio)
Two different candles, P and Q, lighted at the same time were burning at different, but constant, rates. When candle P had burned 16 mm, candle Q had burned 10 mm. When candle Q had burned 35 mm, how many millimeters would candle P have burned?

Nonproportional Problem (Invariant Difference)
Two identical candles, A and B, lighted at different times were burning at the same constant rate. When candle A had burned 20 mm, candle B had burned 12 mm. When candle B had burned 30 mm, how many millimeters would candle A have burned?

Fostering Attendance to Meaning

Students who do not pay attention to the meaning of symbols are more likely to exhibit hammer-and-nail behaviors. For example, students might use a proportion, $a/b = c/x$, to solve a missing-value problem without paying attention to the meaning of each ratio in the context of the problem situation. Consequently, they are more likely to set up a proportion to solve a nonproportional missing-value problem, as in the solution in figure 1a. To foster attendance to meaning in the context of proportional reasoning, students should be challenged to explain what the ratio a/b represents in terms of the problem situation and why the other ratio, c/x, must be equal to a/b.

The proportional problem above can be solved using either $35/10 = x/16$ (i.e., a within-measure proportion) or $16/10 = x/35$ (i.e., a between-measures proportion). The meaning of 35/10 is substantially different from the meaning of 16/10. The within-measure ratio 35/10 is a multiplicative comparison of two quantities; its value of 3.5 means that 35 mm is 3.5 times of 10 mm. The ratio $x/16$ is equal to 3.5 because both candles P and Q are burning at constant rates and lighted at the same time, and the time interval from the first moment (10 mm for candle Q and 16 mm for candle P) to the second moment (35 mm for candle Q and x mm for candle P) is the same for both candles. On the other hand, the between-measure ratio 16/10 is the ratio of candle P's burning rate to candle Q's burning rate, and the invariant value of 1.6 means that candle P is always burning 1.6 times as fast as candle Q because both candles are lighted at the same time and are burning at constant rates. Hence, the ratio 35/x must also equal 1.6. Whereas the value of 1.6 is invariant with respect to time, the ratio of 3.5 depends on the time when the second set of measurements (35 mm for Q and x mm for P) are recorded. In terms of types of division, the ratio 35/10 involves the *how-many-groups* interpretation, or the *measurement model*, whereas 16/10 involves the *how-many-units-in-one-group* interpretation, or the *sharing-equally model*.

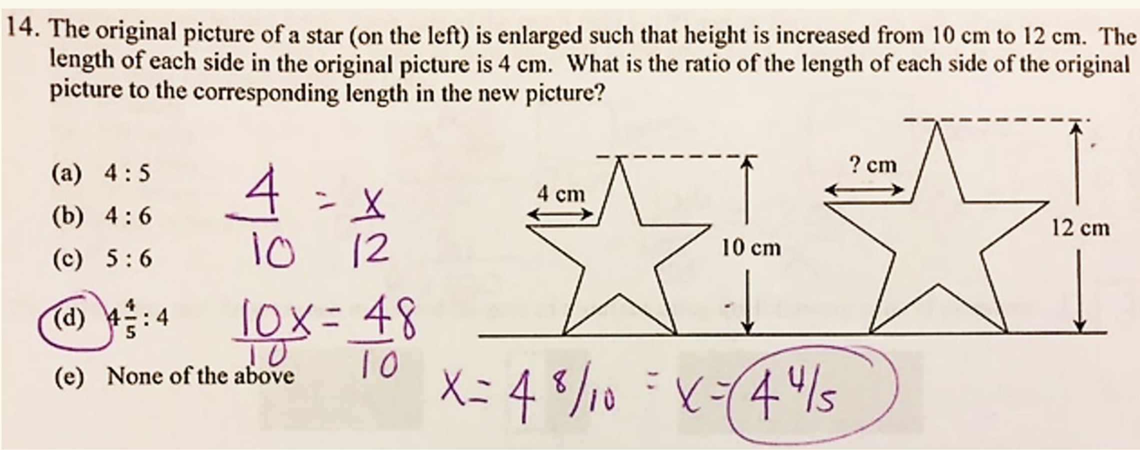
The nonproportional problem emphasizes additive reasoning; it can be solved using $30 - 12 = x - 20$ or using $20 - 12 = x - 30$. Whereas the difference of 18 mm in the first equation refers to an additional amount of burned length between the two moments for candle B, the difference of 8 mm in the second equation refers to the 8 mm head start that candle A has over candle B. The pair of problems offers students an opportunity to attend to the meaning of various values such as 3.5, 1.6, 18, and 8.

Assessing and Developing Conceptual Understanding

Conceptual items can be included in assessments to determine whether students rely on relational understanding or instrumental understanding. Figure 6 shows the work of a middle school teacher who solved the problem by setting up a proportion and solving for x and selecting the answer that contains 4 4/5 without paying attention to the order of the two numbers in the ratio. This problem was designed to assess whether one understands that a proportion implies the equivalence of two ratios. When the star-shaped figure is scaled proportionally, the ratio of each original length to its corresponding enlarged length remains invariant (which is 10:12 or 5:6). Of 20 middle school teachers who answered this question in a professional development workshop setting, only four teachers chose the correct answer "c." Seven teachers chose "d," which is incorrect because the order matters. Six teachers choose "e" because the expected ratio of 4:4 4/5 was not among the answer choices. Two teachers chose "a" after rounding 4 4/5 to 5, and one teacher chose "b," probably because she decided to go with an additive difference of 2 cm. Such error-eliciting problems are a good means for students to learn from their conceptual misunderstandings (Lim 2014). Formative assessment items can be written to assess whether students truly understand a concept or apply formulas without understanding.

Teachers can capitalize on students' solutions and mistakes to generate discussion and foster conceptual understanding. With reference to the solution in figure 1a, students can be pressed to explain what the ratio 30/75 represents and why the ratio 60/x is, or is

Fig. 6



A multiple-choice item like this can assess students' relational understanding of a proportion. Do they understand that a proportion implies the equivalence of two ratios?

not, equal to 30/75. For figure 1b, a teacher may ask the class to discuss how the student made sense of the problem solution and adjusted his solution strategy. Follow-up questions may be posed to have students consider the appropriateness of halving instead of doubling the 75 mm and discuss the difference between an inversely proportional relationship and a negative-slope linear relationship. In discussing the logic underlying the correct solution in figure 1c, students may be challenged to determine the value of an invariant quantity (i.e., the original height of the candle) that is critical for solving this problem.

IN SUMMARY

This article has the potential to create awareness among educators about the hammer-and-nail phenomenon and the danger of spending too much class time on procedures without connections. Pedagogical suggestions are offered for teachers to help students develop a cognitive habit of maintaining *control* over their own mathematical thinking when they solve problems. Problems that are superficially similar but structurally different can elicit student mistakes and create a need for students to appreciate quantitative reasoning, attend to meaning, and make connections. Frequent use of SS-SD problems helps students realize the disadvantages of mathematical impulsivity and the need to be analytic. By becoming situation-oriented, as opposed to tool-oriented, students gain a better understanding of the problem situation and better control over their use of strategies and procedures. When implemented effectively, SS-SD problems can remind students of the inadequacy of instrumental understanding and highlight the importance of relational understanding. A combination of analytic disposition and relational understanding can empower students to overcome their hammer-and-nail tendencies.

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